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Lie symmetries for equations in conformal geometries

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Abstract

We seek exact solutions to the Einstein field equations which arise when two spacetime geometries are conformally related. Whilst this is a simple method to generate new solutions to the field equations, very few such examples have been found in practice. We use the method of Lie analysis of differential equations to obtain new group invariant solutions to conformally related Petrov type D spacetimes. Four cases arise depending on the nature of the Lie symmetry generator. In three cases we are in a position to solve the master field equation in terms of elementary functions. In the fourth case special solutions in terms of Bessel functions are obtained. These solutions contain known models as special cases.

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1. Introduction

Exact solutions, as opposed to computer generated solutions, of the Einstein field equations are of immense importance in understanding the behaviour of a large variety of celestial phenomena. The literature abounds with many different techniques that have been invoked in an effort to obtain new exact solutions for different configurations of matter.

Conformal transformations, as a mathematical procedure, have also been successfully utilized in obtaining new solutions. This is amply illustrated by the comprehensive model of Castejon-Amenedo and Coley (1992). Further, it is known that conformal structures play an important role in twistor theory (Penrose 1999). The restrictive feature of such analyses, however, is the complexity and nonlinearity of the resultant field equations. Some researchers have adopted the Newman–Penrose formalism which, on account of severe integrability constraints arising out of the Bianchi identities, has not proved fruitful in general.

An alternative is to assume the existence of a conformal symmetry on the manifold in order to generate solutions to the Einstein field equations, as these symmetries impose additional restrictions on the metric tensor field and the field equations may be simplified. The physical significance of conformal Killing vectors is that they generate constants of the motion along null geodesics for massless particles.

The maximal spanning G_{15} of conformal motions for Minkowski space is given by Choquet-Bruhat *et al* (1982) and for Robertson–Walker spacetimes by Maartens and Maharaj (1986). In addition the conformal geometries of the *pp*-wave spacetimes (Maartens and Maharaj 1991), static spherically symmetric spacetimes (Maartens *et al* 1995, 1996), Bianchi I and V locally rotationally symmetric spacetimes (Moodley 1992) and Stephani spacetimes (Moopanar 1993) have been completely determined. Kramer (1990) was able to generate a class of metrics for rigidly rotating perfect fluids which admit a proper conformal Killing vector. Additionally, rigidly rotating perfect fluids admitting two Killing vectors and a proper conformal Killing vector were studied by Kramer and Carot (1991). Axisymmetric spacetimes, in the general case of differential rotation, were examined by Mars and Senovilla (1993, 1994).

Several new results, utilizing this conformal symmetry approach, have recently been obtained by Castejon-Amenedo and Coley (1992), Coley and Tupper (1990a, 1990b, 1990c), Dyer *et al* (1987) and Maharaj *et al* (1991). In particular, Maartens and Mellin (1996) used the conformal symmetries of Bianchi I spacetime to demonstrate that the expansion of anisotropic radiation universes tends towards isotropy at late times.

Our interest lies in spacetimes that admit an s-dimensional Lie algebra C_s of conformal motions. We exploit the Defrise-Carter theorem (1975) to generate new models of perfect fluids. We select a spacetime of Petrov type D and its conformally related counterpart in an effort to obtain new solutions to the associated Einstein field equations. We analyse the field equations and show that they can be reduced to a simpler form. The results of Castejon-Amenedo and Coley are regained as a special case of a more general class of exact solutions. Group invariant solutions to a particular field equation which acts as a master equation for the entire system are sought. This analysis generates a rich class of solutions. A number of cases arise depending on the nature of the Lie symmetry generator. In all cases we are in a position to provide solutions to the master field equation in terms of elementary functions. This analysis demonstrates the value of the Lie analysis of differential equations in this application.

2. Spacetime geometry

We consider the line element

$$ds^2 = -dt^2 + dx^2 + e^{2\nu(y,z)}(dy^2 + dz^2) \tag{1}$$

which is of Petrov type D. The spacetime (1) admits three Killing vectors

$$\mathbf{X}_1 = \frac{\partial}{\partial t} \tag{2}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x} \tag{3}$$

$$\mathbf{X}_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \tag{4}$$

which obey the commutation relations

$$[\mathbf{X}_1, \mathbf{X}_2] = 0 \qquad [\mathbf{X}_1, \mathbf{X}_3] = 0 \qquad [\mathbf{X}_2, \mathbf{X}_3] = 0.$$

The Lie algebra of Killing vectors is a G_3 of motions, with the group structure satisfying the above relations.

Suppose that a manifold $(M, \mathbf{g})$ is neither conformally flat nor conformally related to a generalized plane wave. Then according to a theorem due to Defrise-Carter (1975), a Lie algebra of conformal Killing vectors on M with respect to $\mathbf{g}$ can be regarded as a Lie algebra of Killing vectors with regard to some metric on M conformally related to $\mathbf{g}$. Therefore if a spacetime admits the conformal group C_s, then either it is conformally flat ($s = 15$), conformally related to a generalized plane wave ($s \leqslant 7$), or the metric $\bar{g}_{ab} = e^{2U} g_{ab}$ where g_{ab} admits an s-dimensional ($s \leqslant 6$) isometry group. The last possibility is evident above, and so as our starting point we consider the conformally related metric

$$ds^2 = e^{2U}[-dt^2 + dx^2 + e^{2\nu(y,z)}(dy^2 + dz^2)] \tag{5}$$

where $U = U(t, x, y, z)$. The Killing vectors (2)–(4) are now conformal Killing vectors of the conformally related spacetime (5). The Weyl conformal tensor is given by

$$\begin{aligned} 3e^{2\nu}\bar{C}_{0101} &= -6\bar{C}_{0202} = -6\bar{C}_{0303} = 6\bar{C}_{1212} \\ &= 6\bar{C}_{1313} = \tfrac{3}{5}\, e^{-2\nu}\bar{C}_{2323} = \tfrac{3}{5}\, e^{-2\nu}\bar{C}_{3232} = \nu_{yy} + \nu_{zz} \end{aligned}$$

which clearly indicates that the metric (5) is not conformally flat in general. (We use the notation that overhead bars on quantities are defined in the conformally related spacetimes (5).) Note that $\nu_{yy} + \nu_{zz}$ is not zero. We make the assumption that

$$\nu_{yy} + \nu_{zz} = -2k\, e^{2\nu} \tag{6}$$

where k is a nonzero constant. This choice is made on the grounds of simplicity and follows the treatment of Castejon-Amenedo and Coley (1992). In addition if $k = 0$ then the line element (1) becomes flat, and consequently (5) would be conformally flat. Condition (6), with $k \neq 0$, obviates this occurrence.

To determine the perfect fluid energy–momentum tensor, we select a fluid 4-velocity vector $\mathbf{u}$ that is noncomoving with the form

$$u^a = e^{-U}\big(\cosh v\delta_0^a + \sinh v\delta_1^a\big) \tag{7}$$

where $v = v(t, x)$. Note that a trivial calculation reveals that for $v =$ constant, the perfect fluid Einstein field equations would imply conformal flatness.

3. Field equations and kinematics

The Einstein field equations are given by

$$U_t U_y - U_{ty} = 0 \tag{8}$$

$$U_t U_z - U_{tz} = 0 \tag{9}$$

$$U_x U_y - U_{xy} = 0 \tag{10}$$

$$U_x U_z - U_{xz} = 0 \tag{11}$$

$$U_t U_x - U_{tx} = -\tfrac{1}{4}(\mu + p)\, e^{2U} \sinh 2v \tag{12}$$

$$U_y U_z - U_{yz} + \nu_z U_y + \nu_y U_z = 0 \tag{13}$$

$$\begin{aligned} -2U_{xx} - U_x^2 + 3U_t^2 - e^{-2\nu}\big(2U_{yy} + 2U_{zz} + U_y^2 + U_z^2 + \nu_{yy} + \nu_{zz}\big) \\ = (\mu + p)\, e^{2U} \cosh^2 v - p\, e^{2U} \end{aligned} \tag{14}$$

$$\begin{aligned} -2U_{tt} - U_t^2 + 3U_x^2 + e^{-2\nu}\big(2U_{yy} + 2U_{zz} + U_y^2 + U_z^2 + \nu_{yy} + \nu_{zz}\big) \\ = (\mu + p)\, e^{2U} \sinh^2 v + p\, e^{2U} \end{aligned} \tag{15}$$

$$2U_{zz} + U_z^2 + 3U_y^2 + 2\nu_y U_y - 2\nu_z U_z + e^{2\nu}\big(2U_{xx} - 2U_{tt} + U_x^2 - U_t^2\big) = p\, e^{2\nu+2U} \tag{16}$$

$$2U_{yy} + U_y^2 + 3U_z^2 - 2\nu_y U_y + 2\nu_z U_z + e^{2\nu}\big(2U_{xx} - 2U_{tt} + U_x^2 - U_t^2\big) = p\, e^{2\nu+2U} \tag{17}$$

for the line element (5).

The vorticity $\bar{\omega}_{ab}$ is given by

$$\bar{\omega}_{ab} = 0$$

so that the gravitational field is irrotational. The components of the acceleration $\dot{\bar{u}}^a$ are

$$\dot{\bar{u}}_0 = -[(v_x + U_t)\sinh^2 v + (v_t + U_x)\cosh v \sinh v] \tag{18}$$

$$\dot{\bar{u}}_1 = (v_t + U_x)\cosh^2 v + (v_x + U_t)\cosh v \sinh v \tag{19}$$

which are nonzero in general. The expansion $\bar{\Theta}$ is given by the expression

$$\bar{\Theta} = \mathrm{e}^{-U}[(v_t + U_x)\sinh v + (v_x + U_t)\cosh v]. \tag{20}$$

The shear tensor components have the form

$$\bar{\sigma}_{00} = \tfrac{1}{3}[(2\sinh^2 v - 1)\,\mathrm{e}^U + \mathrm{e}^{-U}][(v_t + U_x)\sinh v + (v_x + U_t)\cosh v] \tag{21}$$

$$\bar{\sigma}_{01} = -\tfrac{2}{3}\,\mathrm{e}^U \sinh v \cosh v[(v_t + U_x)\sinh v + (v_x + U_t)\cosh v] \tag{22}$$

$$\bar{\sigma}_{11} = \tfrac{1}{3}[(2\cosh^2 v + 1)\,\mathrm{e}^U - \mathrm{e}^{-U}][(v_t + U_x)\sinh v + (v_x + U_t)\cosh v] \tag{23}$$

$$\bar{\sigma}_{22} = -\tfrac{1}{3}\,\mathrm{e}^{2\nu - U}[(v_t + U_x)\sinh v + (v_x + U_t)\cosh v] = \bar{\sigma}_{33} \tag{24}$$

which do not vanish in general. From (18)–(24) we observe that the fluid congruences of the conformally related line element (5) are accelerating, expanding and shearing.

4. Reduction of the field equations

The most general functional form admitted by equations (8)–(11) is

$$\mathrm{e}^{-U} = f(t, x) + h(y, z) \tag{25}$$

where f and h are arbitrary. The remaining field equations (12)–(17), respectively, then assume the following form:

$$2(f + h)f_{tx} = -(\mu + p)\cosh v \sinh v \tag{26}$$

$$h_{yz} = \nu_z h_y + \nu_y h_z \tag{27}$$

$$-3\big(f_t^2 - f_x^2\big) - 2(f + h)f_{xx} - 2k(f + h)^2 + \mathrm{e}^{-2\nu}\big[3\big(h_y^2 + h_z^2\big) - 2(f + h)(h_{yy} + h_{zz})\big] = -(\mu + p)\cosh^2 v + p \tag{28}$$

$$-3\big(f_t^2 - f_x^2\big) + 2(f + h)f_{tt} - 2k(f + h)^2 + \mathrm{e}^{-2\nu}\big[3\big(h_y^2 + h_z^2\big) - 2(f + h)(h_{yy} + h_{zz})\big] = (\mu + p)\sinh^2 v + p \tag{29}$$

$$-3\big(f_t^2 - f_x^2\big) + 2(f + h)(f_{tt} - f_{xx}) + \mathrm{e}^{-2\nu}\big[3\big(h_y^2 + h_z^2\big) - 2(f + h)(\nu_y h_y - \nu_z h_z + h_{zz})\big] = p \tag{30}$$

$$-3\big(f_t^2 - f_x^2\big) + 2(f + h)(f_{tt} - f_{xx}) + \mathrm{e}^{-2\nu}\big[3\big(h_y^2 + h_z^2\big) - 2(f + h)(\nu_z h_z - \nu_y h_y + h_{yy})\big] = p \tag{31}$$

where the variable U has been replaced by f and h via (25).
The dynamical quantities have the following forms:

$$p = -3\big(f_t^2 - f_x^2\big) + 2(f + h)(f_{tt} - f_{xx}) + \mathrm{e}^{-2\nu}\big(3\big(h_y^2 + h_z^2\big) - (f + h)(h_{yy} + h_{zz})\big) \tag{32}$$

for the pressure p, and

$$\mu = 3\left(f_t^2 - f_x^2\right) + 4k(f+h)^2 - 3\mathrm{e}^{-2\nu}\left(\left(h_y^2 + h_z^2\right) - (f+h)(h_{yy} + h_{zz})\right) \tag{33}$$

for the energy density μ. We now seek an expression for the quantity $v(t, x)$. This is accomplished by first substituting (32) in (28) for p (but we retain the $p\cosh^2 v$ term) to give

$$(\mu + p)\cosh^2 v = 2(f+h)f_{tt} + 2k(f+h)^2 + \mathrm{e}^{-2\nu}(f+h)(h_{yy} + h_{zz}). \tag{34}$$

On substituting (32) into (29) for p (but we retain the $p\sinh^2 v$ term) we get

$$(\mu + p)\sinh^2 v = 2(f+h)f_{xx} - 2k(f+h)^2 - \mathrm{e}^{-2\nu}(f+h)(h_{yy} + h_{zz}). \tag{35}$$

Dividing (35) by (34) yields

$$\tanh^2 v = \frac{2f_{xx} - 2k(f+h) - \mathrm{e}^{-2\nu}(h_{yy} + h_{zz})}{2f_{tt} + 2k(f+h) + \mathrm{e}^{-2\nu}(h_{yy} + h_{zz})}. \tag{36}$$

Another expression for $\tanh v$ may be obtained by dividing (35) with (26) to give

$$\tanh^2 v = \frac{(2f_{xx} - 2k(f+h) - \mathrm{e}^{-2\nu}(h_{yy} + h_{zz}))^2}{4f_{tx}^2}.$$

Comparing this equation with (36) yields the differential equation

$$f_{tx}^2 = (2f_{xx} - 2k(f+h) - \mathrm{e}^{-2\nu}(h_{yy} + h_{zz}))(2f_{tt} + 2k(f+h) + \mathrm{e}^{-2\nu}(h_{yy} + h_{zz})). \tag{37}$$

On rearranging (36) we obtain

$$\frac{2f_{xx} - 2f_{tt}\tanh^2 v}{\tanh^2 v + 1} - 2kf = 2kh + \mathrm{e}^{-2\nu}(h_{yy} + h_{zz})$$

from which it is clear that the left-hand side is a function of t and x whereas the right-hand side is a function of y and z. This implies that the variables separate, and we can put

$$\frac{2f_{xx} - 2f_{tt}\tanh^2 v}{\tanh^2 v + 1} - 2kf = 2kh + \mathrm{e}^{-2\nu}(h_{yy} + h_{zz}) = \alpha$$

where α is a constant. We thus derive the expression

$$h_{yy} + h_{zz} = \mathrm{e}^{2\nu}(\alpha - 2kh). \tag{38}$$

Equation (38) further simplifies the system (26)–(31).

From the above analysis it is clear that the field equations (26)–(31) can be expressed in a simpler form. The resulting system is given by

$$\mu = 3\left(f_t^2 - f_x^2\right) + (f+h)(4kf - 2kh + 3\alpha) - 3\mathrm{e}^{-2\nu}\left(h_y^2 + h_z^2\right) \tag{39}$$

$$p = -3\left(f_t^2 - f_x^2\right) + (f+h)(2f_{tt} - 2f_{xx} + 2kh - \alpha) + 3\mathrm{e}^{-2\nu}\left(h_y^2 + h_z^2\right) \tag{40}$$

$$\tanh^2 v = \frac{2f_{xx} - 2kf - \alpha}{2f_{tt} + 2kf + \alpha} \tag{41}$$

$$f_{tx}^2 = \frac{1}{4}(2f_{xx} - 2kf - \alpha)(2f_{tt} + 2kf + \alpha) \tag{42}$$

$$h_{yz} = \nu_z h_y + \nu_y h_z \tag{43}$$

$$h_{yy} - h_{zz} = 2\nu_y h_y + 2\nu_z h_z. \tag{44}$$

The system (39)–(44), subject to condition (6), namely $\nu_{yy} + \nu_{zz} = -2k\,\mathrm{e}^{2\nu}$, must be solved in order to generate a conformally related perfect fluid model.

Castejon-Amenedo and Coley have considered the case $h =$ constant; this constant may be effectively absorbed into f without any loss of generality, and we can consequently set

$$h = 0.$$

With this value of h the Einstein field equations (39)–(44) reduce to

$$p = -3\left(f_t^2 - f_x^2\right) + 2f(f_{tt} - f_{xx}) \tag{45}$$

$$\mu = 3\left(f_t^2 - f_x^2\right) + 4kf^2 \tag{46}$$

$$v = \tanh^{-1}\left(\frac{kf - f_{xx}}{f_{tx}}\right) \tag{47}$$

$$f_{tx}^2 = (f_{tt} + kf)(f_{xx} - kf) \tag{48}$$

which is in agreement with the equations used by Castejon-Amenedo and Coley (1992). Note that (45)–(48) corresponds to $h = \alpha = 0$. It is possible in (39)–(44) to have $\alpha = 0$ with $h \neq 0$. This naturally leads to two categories of exact solutions that we now present.

5. An extension of the Castejon-Amenedo and Coley solutions: $h = 0$

This category of solutions corresponds to $h = \alpha = 0$. In order to generate a solution to the system (45)–(48), it is sufficient to obtain a form for $f = f(t, x)$; in other words (48) must be integrated. To obtain a solution, Castejon-Amenedo and Coley have assumed that the variables separate and set

$$f(t, x) = F(t)G(x) \tag{49}$$

which yields the differential equations

$$F\ddot{F} - \gamma\dot{F}^2 + kF^2 = 0 \tag{50}$$

$$GG'' - \frac{1}{\gamma}G'^2 - kG^2 = 0 \tag{51}$$

where γ is a constant.

5.1. Case I. $\gamma = 1$

It is easy to solve equations (50), (51) in general. As a result, the system (45)–(47) reduces to

$$\mu = \mathrm{e}^{k(x^2-t^2)+2k_1t+2k_2x}\left[3k^2(t^2 - x^2) - 6k(k_1t + k_2x) + 3\left(k_1^2 - k_2^2\right) + 4k\right] \tag{52}$$

$$p = \mathrm{e}^{k(x^2-t^2)+2k_1t+2k_2x}\left[-k^2(t^2 - x^2) + 2k(k_1t + k_2x) - \left(k_1^2 - k_2^2\right) - 4k\right] \tag{53}$$

$$\tanh v = \frac{kx + k_2}{kt - k_1}. \tag{54}$$

This general solution contains the solution of Castejon-Amenedo and Coley when we set

$$k_1 = k_2 = 0$$

in (52)–(54). For this choice of constants we obtain

$$\mu = \mathrm{e}^{k(x^2-t^2)}[3k^2(t^2 - x^2) + 4k] \tag{55}$$

$$p = \mathrm{e}^{k(x^2-t^2)}[k^2(x^2 - t^2) - 4k] \tag{56}$$

$$\tanh v = \frac{x}{t} \tag{57}$$

which is the exact solution of Castejon-Amenedo and Coley (1992). Consequently, we have extended their solution (55)–(57) to the more general class (52)–(54).
For the exact solution (52)–(54) we obtain

$$\mu + p = 2\mathrm{e}^{k(x^2-t^2)+2k_1t+2k_2x}\left[k^2(t^2-x^2) - 2k(k_1t+k_2x) + \left(k_1^2-k_2^2\right)\right]$$
$$\mu + 3p = -8k\,\mathrm{e}^{k(x^2-t^2)+2k_1t+2k_2x}$$
$$\mu - p = 4\mathrm{e}^{k(x^2-t^2)+2k_1t+2k_2x}\left[k^2(t^2-x^2) - 2k(k_1t+k_2x) + \left(k_1^2-k_2^2\right) + 2k\right]$$

and it is possible to study the weak, dominant and strong energy conditions. We note that the appearance of the constants k_1 and k_2 allows for a wider range of behaviour for our class of solutions than is the case with the Castejon-Amenedo and Coley (1992) exact solution. For this model $\dot{\bar{u}}^a = 0$ so that the field is nonaccelerating; however both $\bar{\sigma}_{ab}$ and $\bar{\Theta}$ are nonzero which implies that the gravitational field is shearing and expanding.

5.2. *Case II.* $\gamma \neq 1$

The case with $\gamma \neq 1$ in (45) is of interest in generating cosmological models for investigating a wider range of physical behaviour. The transformations

$$F = u_1^{\frac{1}{1-\gamma}}, \qquad G = u_2^{\frac{\gamma}{\gamma-1}}$$

reduce equations (50)–(51) to

$$\ddot{u}_1 + k(1-\gamma)u_1 = 0 \qquad u_2'' + k\frac{(1-\gamma)}{\gamma}u_2 = 0$$

which are linear in u_1 and u_2 respectively. The solutions of this system of differential equations are easily obtained in terms of elementary functions, and consequently we do not present their explicit forms here.

6. Group invariant solutions: $h \neq 0$

This category of solutions corresponds to $h \neq 0$ with $\alpha = 0$. It is possible to solve the more general system (39)–(44) with $h \neq 0$. If we take $h = h(y)$, then (43) implies that $\nu = \nu(y)$. Consequently, condition (6) becomes the ordinary differential equation

$$\nu_{yy} = -2k\,\mathrm{e}^{2\nu}. \tag{58}$$

Equation (44) reduces to

$$h_y = C\,\mathrm{e}^{2\nu} \tag{59}$$

where C is a constant of integration. The general solution to (58) and (59) is given by

$$\mathrm{e}^{2\nu} = -\frac{A}{k}\,\mathrm{cosech}^2(\sqrt{2A}y + B) \tag{60}$$

$$h(y) = \frac{C\sqrt{A}}{\sqrt{2}k}\coth(\sqrt{2A}y + B) + D \tag{61}$$

where A, B and D are constants of integration. In order to complete the solution, we need to solve (42) and obtain f as this immediately yields expressions for p, μ and v. We now focus our attention on (42) with $\alpha = 0$, i.e.

$$f_{tx}^2 = (f_{tt} + kf)(f_{xx} - kf) \tag{62}$$

with the objective of generating solutions in a systematic manner. We achieve this by invoking the method of Lie group analysis of differential equations (Olver 1993). The Lie symmetry generators of (62) are found with the help of the computer package PROGRAM LIE (Head 1993). The infinitesimal Lie symmetry generators for the partial differential equation (62) are

$$Z_1 = \frac{\partial}{\partial t} \tag{63}$$

$$Z_2 = \frac{\partial}{\partial x} \tag{64}$$

$$Z_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \tag{65}$$

$$Z_4 = f\frac{\partial}{\partial f}. \tag{66}$$

It is clear that these symmetries form the Lie algebra $A_{3,4} \oplus A_1$ (Patera and Winternitz 1977) with basis given by

$$G_1 = Z_1 + Z_2 \tag{67}$$

$$G_2 = Z_1 - Z_2 \tag{68}$$

$$G_3 = Z_3 \tag{69}$$

$$G_4 = Z_4. \tag{70}$$

It is this basis that we will use in our subsequent analysis. Before proceeding further, we observe that (62) is invariant under the following discrete transformations:

$$f \to -f \tag{71}$$

$$t \to -t \tag{72}$$

and

$$x \to -x. \tag{73}$$

Taking these reflections into account, we have

$$G_4 \to -G_4 \tag{74}$$

$$G_2 \to -G_1 \tag{75}$$

and

$$G_3 \to -G_3. \tag{76}$$

We also note that $x \to -x$ will make $G_1 = G_2$.

In order to obtain group invariant solutions of (62) (with $f = f(t, x)$ explicitly), we only need to consider the following symmetry combinations (Msomi 2003):

$$G_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \tag{77}$$

$$G_4 = f\frac{\partial}{\partial f} \tag{78}$$

$$G_1 + G_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + f\frac{\partial}{\partial f} \tag{79}$$

$$G_3 + \beta G_4 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + \beta f\frac{\partial}{\partial f} \tag{80}$$

$$G_1 + G_2 + \beta G_4 = 2\frac{\partial}{\partial t} + \beta f\frac{\partial}{\partial f} \tag{81}$$

where β is a real arbitrary parameter. All other solutions of (62) obtained via linear combinations of the point symmetries (63)–(66) can be obtained from the solutions we report here. We consider each in turn.

6.1. Invariance under G_1

Using G_1 we determine the invariants from the invariant surface condition

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}f}{0} \tag{82}$$

which yields

$$y = x - t \tag{83}$$

$$f = V. \tag{84}$$

Now, by using the above transformation in equation (62), the partial differential equation is reduced to

$$k^2 V^2 = 0 \tag{85}$$

i.e.

$$V = 0. \tag{86}$$

Thus only trivial travelling wave solutions are possible.

6.2. Invariance under G_4

In this case the system has invariants given by

$$y = t \tag{87}$$

$$V = x. \tag{88}$$

From the above transformation, we cannot reduce (62). There is no ordinary differential equation to solve.

6.3. Invariance under $G_1 + G_4$

Here, the invariants are:

$$y = x - t \tag{89}$$

$$f = V\,\mathrm{e}^t. \tag{90}$$

The partial differential equation is reduced by the transformation to the form

$$\big(k(1+k)V^2 + V_y^2 - V(2KV_y + V_{yy})\big) = 0 \tag{91}$$

which is linearizable and has solution

$$\log V = \frac{(1+k)}{2}y - \frac{(1+k)}{4k} + \frac{C_0}{2k} + C_1\,\mathrm{e}^{-2ky} \tag{92}$$

where C_0 and C_1 are arbitrary constants, whence

$$f = \exp\left(\frac{(1+k)}{2}x + \frac{(1-k)}{2}t - \frac{(1+k)}{4k} + \frac{C_0}{2k} + C_1\,\mathrm{e}^{-2k(x-t)}\right) \tag{93}$$

and so we have 'time-boosted' travelling wave solutions. With h given by (61) and f by (93) it is possible to show that there are regions for which $\mu > 0$ and $p > 0$. This is a desirable feature because we expect that barotropic matter in cosmological models should have positive pressures and positive energy densities.

6.4. *Invariance under* $G_3 + \beta G_4$

For this case the system has invariants given by

$$y = x^2 - t^2 \tag{94}$$
$$f = (x+t)^\beta V \tag{95}$$

which reduces the partial differential equation to

$$k^2V^2 + 4(V_y(1+\beta))^2 + 8yV_yV_{yy} - 4VV_yk(1+\beta) - 4VV_{yy}ky + 4VV_{yy}\beta(1-\beta) = 0. \tag{96}$$

This equation is an ordinary differential equation with the single symmetry

$$Z_1 = V\frac{\partial}{\partial V}. \tag{97}$$

In order to solve the differential equation (96) with one symmetry, we first try to reduce the order of this equation and see if the resulting equation is easily integrated.

The differential equation (96) has the following reduction variables from (97):

$$r = y \tag{98}$$
$$q = \frac{V_y}{V}. \tag{99}$$

Using these invariants in (96), we have

$$\frac{\mathrm{d}q}{\mathrm{d}r} = \frac{-k^2 - 4q^2(1+\beta)^2 + 4q(1+\beta)}{8pq - 4qkp - 4(-1+\beta)\beta)} - q^2. \tag{100}$$

In this case, we find that the first-order equation cannot be easily integrated. We would have to resort to numerical solutions. However, some special solutions can still be found, as demonstrated later.

6.5. *Invariance under* $G_1 + G_2 + \beta G_4$

In this final case the invariants are

$$y = x \tag{101}$$
$$f = V\mathrm{e}^{\frac{t\beta}{2}}. \tag{102}$$

The partial differential equation is reduced by this transformation to the form

$$(\beta^2 + 4k)VV_{yy} - \beta^2V_y^2 - k(\beta^2+4k)V^2 = 0 \tag{103}$$

which has general solution

$$V = 2^{\frac{\beta^2+4k}{4k}}\left(-\frac{(C_1\sin\psi - C_2\cos\psi)^2}{(\beta^2+4k)}\right)^{\frac{\beta^2+4k}{8k}} \tag{104}$$

where C_1 and C_2 are arbitrary constants and

$$\psi = \frac{2kx}{\sqrt{-\beta^2-4k}}. \tag{105}$$

Thus a solution to (62) is given by

$$f = 2^{\frac{\beta^2+4k}{4k}}\mathrm{e}^{\frac{t\beta}{2}}\left(-\frac{(C_1\sin\psi - C_2\cos\psi)^2}{(\beta^2+4k)}\right)^{\frac{\beta^2+4k}{8k}}. \tag{106}$$

7. A particular solution

In order to investigate the physical properties of the spacetimes obtained, we take a special case of the intractable result in section 6.4, that of $\beta = 0$. In this case the new independent variable is given by

$$u = t^2 - x^2 \tag{107}$$

and the new functional form is

$$f(t, x) = f(u) = f(t^2 - x^2). \tag{108}$$

As a result, (62) becomes

$$4u\frac{d^2 f}{du^2} + 2\frac{df}{du} + kf = 0. \tag{109}$$

We now define new variables a and q via

$$f(u) = a(u)u^{1/4} \qquad q = u^{1/2} \tag{110}$$

and obtain the equation

$$q^2\frac{d^2 a}{dq^2} + q\frac{da}{dq} + \left(kq^2 - \left(\frac{1}{2}\right)^2\right)a = 0. \tag{111}$$

We now distinguish between the two cases $k > 0$ and $k < 0$.

For $k > 0$, (111) can be simplified to

$$w^2\frac{d^2 a}{dw^2} + w\frac{da}{dw} + \left(w^2 - \left(\frac{1}{2}\right)^2\right)a = 0 \tag{112}$$

by the transformation $\sqrt{kq} = w$. Equation (112) is the Bessel equation of order $\frac{1}{2}$, and its solutions are the linearly independent Bessel functions $J_{\frac{1}{2}}(w)$ and $J_{-\frac{1}{2}}(w)$ which may be expressed in terms of elementary trigonometric functions in the following way:

$$J_{\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}}\sin w \qquad J_{-\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}}\cos w.$$

It is a pleasing feature of this model that (62) admits Bessel functions as solutions since many realistic phenomena are governed by the Bessel equation. The solution to the differential equation (62), in the original variables t and x, is

$$f(x, t) = A\sin\sqrt{k(t^2 - x^2)} + B\cos\sqrt{k(t^2 - x^2)} \tag{113}$$

where A and B are arbitrary constants. For the solution (113) the quantity v is given by

$$\tanh v = \frac{t}{x}$$

so that the exact solution corresponding to (113) is not conformally flat.

For $k < 0$, we make the substitution $k = -\beta^2$ followed by $W = \beta z$. Equation (111) then has the form

$$W^2\frac{d^2 a}{dW^2} + W\frac{da}{dW} - \left(W^2 + \left(\frac{1}{2}\right)^2\right) = 0$$

which is the modified Bessel equation of order $\frac{1}{2}$. This differential equation admits the modified Bessel functions

$$I_{\frac{1}{2}} = \sqrt{\frac{2}{\pi W}}\sinh W \tag{114}$$

$$I_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi W}} \cosh W \tag{115}$$

as linearly independent solutions. The solution to the field equation (62), in terms of the original variables t and x, may be written as

$$f(x,t) = A \sinh \sqrt{k(t^2 - x^2)} + B \cosh \sqrt{k(t^2 - x^2)} \tag{116}$$

analogous to (113).

8. Conclusion

The notion of conformally mapping a given line element to a new metric, such as the conformally related metric (5), is potentially a very fertile avenue in generating new solutions to the Einstein field equations. However, the actual number of exact solutions found using this algorithm is very low as pointed out by Stephani *et al* (2003). The Petrov type D spacetime turns out to be a rare metric which is amenable to this approach, and we have therefore focused our attention on this line element (first considered by Castejon-Amenedo and Coley (1992)). The Petrov type D model investigated is physically well behaved as there is a barotropic equation of state; the weak and strong energy conditions are satisfied and it is amendable to a simple two perfect fluid interpretation.

In order to demonstrate a new solution we had to find new functions h and f which define the conformal function U in (5). Two categories of solutions arise naturally in the analysis. In the first category ($h = 0$) we found a general class of solutions in terms of elementary functions. This contains the solutions of Castejon-Amenedo and Coley (1992) as special cases. In the second category ($h \neq 0$) we invoked the method of Lie group analysis to solve a master field equation which allowed us to obtain various exact solutions for the metric under consideration. Four cases arise depending on the Lie symmetry generators (we ignore the non-reducible case). Three of the cases were readily solvable, in general, in terms of elementary functions. However, we were only able to provide special solutions in the remaining case.

This was analysed further for physical plausibility by considering a particular solution. We demonstrated that in this case that the solution can be expressed in terms of Bessel and modified Bessel functions. The positivity of pressure and energy density in this case ensured its relevance for the description of some cosmological processes. All solutions found via the Lie method could be used in both $h = 0$ and $h \neq 0$ cases. This treatment demonstrates the importance of the method of Lie group analysis in seeking solutions to the Einstein field equations.

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